

# Correction to “The classification of the surfaces with parallel mean curvature vector in two-dimensional complex space forms”

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Abstract: We give a condition under which the findings of the paper cited above work well and determine the surfaces that were not considered before.

## 1 Correction

Kenmotsu and Zhou [6] claimed that the surfaces stated in the title were locally classified. The proof used the system of ordinary differential equations in Ogata [7], but in that paper [7], a mistake was made, as pointed out by Hirakawa [4]. In fact, the claim that “ $\lambda$  is a real-valued function defined on  $U$ ” made in line 3 of page 401 in [7] is not correct in general. Hence, our derivation [6] needs an additional assumption:  $\lambda = \bar{\lambda}$ . We follow the notation used in [6] and [7]. In [4], Hirakawa classified the surfaces with  $a = \bar{a}$ , which is weaker than  $\lambda = \bar{\lambda}$ . Hence, we were forced to study the case of  $a \neq \bar{a}$  for the classification of these surfaces. In this paper, we prove the following:

**Correction** *A parallel mean curvature vector surface in a complex two-dimensional complex space form with  $a \neq \bar{a}$  depends on one real-valued harmonic function on the surface and five real constants if the ambient space is not flat, the mean curvature vector does not vanish, and the Kaehler angle is not constant.*

We remark that Hirakawa [4] proved the above result under an additional

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condition  $c \equiv 0$  (for  $c$  see §2). As a by-product of the correction, we have the following corollary:

**Corollary** *Any two-dimensional smooth manifold can be locally embedded in the complex projective plane and in the complex hyperbolic plane as a parallel mean curvature vector surface.*

Some comments about the assumptions of the correction are in order. Chern and Wolfson [2] and Eschenburg, Guadalupe, and Tribuzy [3] studied the case of a zero mean curvature vector, i.e., minimal surfaces. The case of a constant Kaehler angle is included in Hirakawa [4], because the Gaussian curvature of such a surface is constant. Parallel mean curvature vector surfaces in a flat complex space form are locally classified independently by Hoffman [5], Yau [8], and Chen [1], because the ambient space is locally isometric to  $R^4$  with the standard metric.

## 2 Analysis of the structure equations

To correct our calculation, we rewrite the structure equations of the immersion. Let  $\overline{M}[4\rho]$  be a complex two-dimensional complex space form with a constant holomorphic sectional curvature  $4\rho$ ,  $M$  be an oriented and connected real two-dimensional Riemannian manifold with Gaussian curvature  $K$ , and  $x : M \rightarrow \overline{M}[4\rho]$  be an isometric immersion with Kaehler angle  $\alpha$  such that the mean curvature vector field  $H$  is nonzero and parallel for the normal connection on the normal bundle of the immersion. In this section, we show that the Kaehler angle plays an important role in understanding the immersion.

Since all calculations and formulas in [7] are valid until page 400, there exists a local field of unitary coframes  $\{w_1, w_2\}$  on  $\overline{M}[4\rho]$  such that, by restricting it to  $x$ , the Riemannian metric  $ds^2$  on  $M$  is written as  $ds^2 = \phi\bar{\phi}$ , where  $\phi = \cos \alpha/2 \cdot \omega_1 + \sin \alpha/2 \cdot \bar{\omega}_2$ . Let  $a$  and  $c$  be the complexified second fundamental tensors of  $x$  with respect to  $\{\omega_1, \omega_2\}$ . Then, the Kaehler angle  $\alpha$  and the complex 1-form  $\phi$  satisfy

$$d\alpha = (a + b)\phi + (\bar{a} + b)\bar{\phi}, \quad (2.1)$$

$$d\phi = (\bar{a} - b) \cot \alpha \cdot \phi \wedge \bar{\phi}, \quad (2.2)$$

where  $2b = |H| > 0$ . By (2.4), (2.5), and (2.6) of [7], the Gauss, Codazzi-

Mainardi, and Ricci equations of  $x$  are, respectively,

$$K = -4(|a|^2 - b^2) + 6\rho \cos^2 \alpha, \quad (2.3)$$

$$da \wedge \phi = - \left( 2a(\bar{a} - b) \cot \alpha + \frac{3}{2} \rho \sin \alpha \cos \alpha \right) \phi \wedge \bar{\phi}, \quad (2.4)$$

$$dc \wedge \bar{\phi} = 2c(a - b) \cot \alpha \cdot \phi \wedge \bar{\phi}, \quad (2.5)$$

$$|c|^2 = |a|^2 + \frac{\rho}{2}(-2 + 3 \sin^2 \alpha). \quad (2.6)$$

Conversely, for a real number  $\rho$ , a positive number  $b$ , a complex-valued 1-form  $\phi$ , a real-valued function  $\alpha$ , complex-valued functions  $a$  and  $c$  on a simply connected domain  $D$  in  $R^2$ , if they satisfy (2.1)–(2.6), where  $K$  denotes the Gaussian curvature of the Riemannian metric  $ds^2 = \phi \bar{\phi}$  on  $D$ , then there exists an isometric immersion  $x : (D, ds^2) \rightarrow \overline{M}[4\rho]$  such that the Kaehler angle of  $x$  is  $\alpha$ , the mean curvature vector of  $x$  is parallel, and its length is equal to  $2b$ .

The immersion  $x$  is called a general type if it satisfies  $d\alpha \neq 0$ ,  $a \neq \bar{a}$ , and  $c \neq 0$  on  $M$ . From now on, we suppose that the ambient space is nonflat and study the immersions of a general type. We remark that the results in this paper are local in nature, and we suppose that all formulas hold in a neighborhood of  $M$ .

We define  $a_1$  and  $c_1$  by

$$\begin{cases} da = (a_1 + at_1)\phi + t_2\bar{\phi}, \\ dc = 2c(a - b) \cot \alpha \cdot \phi + (c_1 + c\bar{t}_1)\bar{\phi}, \end{cases} \quad (2.7)$$

where  $t_i = t_i(\alpha, a, \bar{a})$  ( $i = 1, 2, \dots$ ) are functions of  $\alpha$ ,  $a$ , and  $\bar{a}$  listed in the Appendix.

Taking the exterior differentiation of (2.6), we have

$$c\bar{c}_1 = \bar{a}a_1. \quad (2.8)$$

The exterior differentiation of (2.7) allows us to define  $a_{11}$  and  $c_{11}$  by

$$\begin{cases} da_1 = a_{11}\phi + (3a_1(\bar{a} - b) \cot \alpha + t_3)\bar{\phi}, \\ dc_1 = (3c_1(a - b) \cot \alpha + ct_4)\phi + c_{11}\bar{\phi}. \end{cases} \quad (2.9)$$

The exterior differentiation of (2.8) implies, by (2.7) and (2.9),

$$|c_1|^2 = |a_1|^2 + t_5. \quad (2.10)$$

By (2.8), we have  $|c|^2|c_1|^2 = |a|^2|a_1|^2$ . This with (2.6) and (2.10) implies

$$\frac{\rho}{2}(-2 + 3\sin^2 \alpha)|a_1|^2 = -(|a|^2 + \frac{\rho}{2}(-2 + 3\sin^2 \alpha)t_5).$$

Hence, if  $\rho \neq 0$  and  $\sin^2 \alpha \neq 2/3$ , then

$$|a_1|^2 = t_6. \quad (2.11)$$

We note that  $a_1 \neq 0$ , since  $x$  is of a general type. Indeed, if  $a_1$  vanishes identically, then so do  $c_1$ , and hence,  $t_4$ . However, by the definition of  $t_4$ , we calculate  $t_4 - \overline{t_4} = -6b(a - \bar{a})(5 + 3\cos 2\alpha)\sin^2 \alpha / (1 + 3\cos 2\alpha)^2$ , which gives a contradiction.

Taking the exterior differentiation of (2.11), we have

$$a_{11} = \frac{t_7 a_1 + t_8}{\overline{a_1}}. \quad (2.12)$$

Inserting this into the first equation of (2.9), we take the exterior differentiation of the formula, which gives us

$$t_9 a_1 + \overline{t_9 a_1} + t_{10} = 0. \quad (2.13)$$

We study first the case of  $t_9 \neq 0$ . By (2.11) and (2.13),  $a_1$  is determined in terms of  $\alpha$ ,  $a$ , and  $\bar{a}$ , denoted  $a_1 = t_{11}(\alpha, a, \bar{a})$ . Let us take the exterior differentiation of this formula. Then, by (2.9) and (2.12) we have two equations for  $\alpha$ ,  $a$ , and  $\bar{a}$ , denoted  $t_{12}(\alpha, a, \bar{a}) = 0$  and  $t_{13}(\alpha, a, \bar{a}) = 0$ . When  $t_9(\alpha, a, \bar{a}) = 0$ , we have  $t_{10}(\alpha, a, \bar{a}) = 0$  by (2.13). These are all relations for  $\alpha$ ,  $a$ , and  $\bar{a}$  that we can get by exterior differentiations of the structure equations. We show the following lemma:

**Lemma 1** *If  $x$  is of a general type, then  $d\alpha \wedge da = 0$ .*

*Proof.* First, we prove Lemma 1 when  $t_9 \neq 0$ . It holds that  $\partial t_{12}/\partial a \neq 0$  or  $\partial t_{12}/\partial \bar{a} \neq 0$ . Indeed, if both are zero, then  $t_{12}$  is a function of  $\alpha$  and then  $t_{12} = 0$  implies that  $\alpha$  is constant, giving a contradiction. We suppose that  $\partial t_{12}/\partial \bar{a} \neq 0$ . Then,  $t_{12}(\alpha, a, \bar{a}) = 0$  can be solved as  $\bar{a} = \bar{a}(\alpha, a)$  by the implicit function theorem. Inserting this into the expression for  $t_{13}$  gives  $t_{13}(\alpha, a, \bar{a}(\alpha, a)) = 0$ . This proves Lemma 1. When  $\partial t_{12}/\partial a \neq 0$ , we prove Lemma 1 in the same way as before, because  $\alpha$  is real valued. In the case of  $t_9 = 0$  also, we can prove Lemma 1 in the same manner as before. Indeed,  $t_{12}$  and  $t_{13}$  now replace  $t_9$  and  $t_{10}$ . This completes the proof of Lemma 1.

Last, we study  $c$  in terms of  $\alpha$ ,  $a$ , and  $\phi$ . From (2.7) and (2.8), we have

$$d \log c = 2(a - b) \cot \alpha \cdot \phi + \left( \frac{c_1}{c} + \bar{t}_1 \right) \bar{\phi}, \quad (2.14)$$

where

$$\frac{c_1}{c} = \frac{\bar{c}c_1}{\bar{c}c} = \frac{a\bar{t}_{11}(\alpha, a, \bar{a})}{|a|^2 + \rho/2(-2 + 3 \sin^2 \alpha)}. \quad (2.15)$$

Since its length  $|c|$  is uniquely determined by (2.6), we have shown the following:

**Lemma 2**  *$c$  is determined up to a real constant by integration of the formula (2.14).*

### 3 Proof of the correction

If  $\alpha$  is not constant, then  $a$  is a function of  $\alpha$ , as given by  $a = a(\alpha)$ , from Lemma 1. It follows from (2.1) and (2.7) that

$$\frac{da}{d\alpha} = \frac{\cot \alpha}{\overline{a(\alpha)} + b} \left( -2ba(\alpha) + 2|a(\alpha)|^2 + \frac{3\rho}{2} \sin^2 \alpha \right), \quad (a + b \neq 0). \quad (3.1)$$

The complex-valued function  $a = a(\alpha)$  is determined by the above first-order ordinary differential equation; hence, it depends on one complex constant. Next, we determine the Kaehler angle  $\alpha$  and the Riemannian metric  $ds^2$  on  $M$ . Let  $\phi = \lambda(z, \bar{z})dz$ , where  $z$  is an isothermal coordinate on  $M$ . From (2.1), we have  $\alpha_z = \lambda(a + b)$ ; hence  $a + b \neq 0$ . Taking its exterior derivative, (2.7) implies  $\alpha_{z\bar{z}} = \lambda_{\bar{z}}(a + b) + \lambda\bar{\lambda}t_2$ . From (2.2), we see that  $\lambda_{\bar{z}} = -(\bar{a} - b)\lambda\bar{\lambda} \cot \alpha$ . From these formulas, we have

$$\alpha_{z\bar{z}} - F(\alpha)\alpha_z\alpha_{\bar{z}} = 0, \quad (3.2)$$

where

$$F(\alpha) = \frac{((a(\alpha) - b)(\overline{a(\alpha)} - b) + 3\rho/2 \sin^2 \alpha)}{(a(\alpha) + b)(\overline{a(\alpha)} + b)} \cot \alpha.$$

**Lemma 3** *Any solution  $\alpha$  of (3.2) is written as  $\alpha(z, \bar{z}) = \psi(f(z, \bar{z}))$ , where  $f(z, \bar{z})$  is a real-valued harmonic function on  $M$  and  $\psi$  is a real solution of the second-order ordinary differential equation*

$$\psi''(t) - F(\psi)\psi'(t)^2 = 0. \quad (3.3)$$

Proof. Define a real valued function  $K(t)$  of one real variable by

$$K(t) = \int e^{-\int F(t)dt} dt$$

and set  $f(z, \bar{z}) = K(\alpha(z, \bar{z}))$ . By (3.2),  $f(z, \bar{z})$  is a harmonic function, i.e.,  $f$  satisfies  $\partial^2 f / \partial z \partial \bar{z} = 0$ . We set  $\psi(t) = K^{-1}(t)$ . Then,  $\psi(t)$  satisfies (3.3), proving Lemma 3.

Because  $\psi(t)$  depends on two real constants, the Kaehler angle  $\alpha$  is determined by one harmonic function on  $M$  and two real constants. By (2.1), the Riemannian metric  $ds^2$  is written as

$$ds^2 = \left| \frac{\alpha_z}{a(\alpha) + b} \right|^2 dz d\bar{z}.$$

In view of Lemmas 1 and 2, we proved the following:

**Theorem 1** *Let  $x : M \rightarrow \overline{M}[4\rho]$  ( $\rho \neq 0$ ) be a parallel mean curvature vector immersion such that  $H \neq 0$ . If  $x$  is of a general type, then it is determined by one real-valued harmonic function on  $M$  and five real constants up to isometries of  $\overline{M}[4\rho]$ .*

Now we discuss the converse of the above and prove the existence of the immersion of a general type. Given  $\rho \neq 0$  and  $b > 0$ , we take a solution  $a = a(\alpha)$  of (3.1). For any real-valued harmonic function  $f(z, \bar{z})$  on a simply connected domain  $D \subset R^2$  with  $f_z \neq 0$  and a solution  $\psi$  of (3.3), we define  $\alpha = \psi(f(z, \bar{z}))$ . We set  $\lambda = \alpha_z / (a(\alpha) + b)$ ,  $\phi = \lambda dz$ , and  $ds^2 = \phi \bar{\phi}$ . It holds that  $d\phi = (\bar{a} - b) \cot \alpha \cdot \phi \wedge \bar{\phi}$ . These  $\alpha$ ,  $a = a(\alpha)$ ,  $\phi$ , and  $ds^2$  satisfy (2.1)–(2.4).  $a_1$  is defined by (2.7); hence, we have

$$a_1 = -a(\alpha)t_1(\alpha, a(\alpha), \overline{a(\alpha)}) + (a(\alpha) + b)\frac{da}{d\alpha}.$$

We determine  $c$  explicitly as follows: We set

$$\omega = \frac{1}{2i} \cdot \frac{\omega_1 \phi - \overline{\omega_1 \phi}}{|a(\alpha)|^2 + \rho/2(-2 + 3 \sin^2 \alpha)},$$

where

$$\begin{aligned} \omega_1 = & \left( |a(\alpha)|^2 + \frac{\rho}{2}(-2 + 3 \sin^2 \alpha) \right) \left( 2(a(\alpha) - b) \cot \alpha - \overline{t_1(\alpha, a(\alpha), \overline{a(\alpha)})} \right) \\ & + \overline{a(\alpha)} \left( a(\alpha)t_1(\alpha, a(\alpha), \overline{a(\alpha)}) - (a(\alpha) + b)\frac{da}{d\alpha} \right). \end{aligned}$$

Then, we have the following:

**Lemma 4**  $\omega$  is a closed real 1-form on  $D$ .

We shall omit the proof, because this is verified by direct calculation using (2.1), (2.2), (2.4), and (3.1).

According to Lemma 4, there exists a real-valued function  $\nu = \nu(z, \bar{z})$  on  $D$  such that  $d\nu = \omega$ . Set

$$c = (|a(\alpha)|^2 + \frac{\rho}{2}(-2 + 3 \sin^2 \alpha)^{1/2} e^{i\nu(z, \bar{z})}). \quad (3.4)$$

Since this satisfies (2.5) and (2.6), these  $\alpha$ ,  $a$ ,  $\phi$ , and  $c$  satisfy the structure equations (2.1)–(2.6). We have thus proved the following:

**Theorem 2** *Given  $\rho \neq 0$ ,  $b > 0$ , and a nonconstant real-valued harmonic function  $f$  on a simply connected domain  $D$  in  $R^2$ , there exist a Riemannian metric  $ds^2$  on  $D$  and a parallel mean curvature vector immersion from  $D$  into  $\overline{M}[4\rho]$  of a general type such that  $|H| = 2b$ , and the Kaehler angle is determined by  $f$ .*

The correction is proved by Theorems 1 and 2.

**Proof of Corollary** Let  $(u, v)$  be a local coordinate of a two-dimensional smooth manifold  $M$ . Then, the coordinates are nonconstant harmonic functions on the neighborhood of  $M$ . Hence, we prove the corollary by setting  $f = u$  or  $f = v$  in Theorem 2.

## 4 Associated family

We show the explicit example of the second fundamental tensors of the surfaces with  $a \neq \bar{a}$  and  $c \neq 0$ . For real numbers  $c_1, c_2$ , we set

$$\begin{aligned} a(t) &= \frac{-4 + (9 + 4c_1) \sin^2 t - 9c_1 \sin^4 t + \sqrt{-1} \sqrt{2(8 - 9 \sin^2 t)(-1 + c_1 \sin^2 t)}}{4(-1 + c_1 \sin^2 t) - \sqrt{-1} \sqrt{2(8 - 9 \sin^2 t)(-1 + c_1 \sin^2 t)}}, \\ \xi(t) &= 2^{5/2} \int \frac{\cot t}{\sqrt{(8 - 9 \sin^2 t)(-1 + c_1 \sin^2 t)}} dt + c_2, \\ c(t) &= \sqrt{\frac{c_1}{2(-9 + 8c_1)}} (8 - 9 \sin^2 t) e^{\sqrt{-1} \xi(t)}. \end{aligned}$$

Given a nonconstant real-valued harmonic function  $f(z, \bar{z})$  on a domain  $D$  in  $R^2$ , we set  $\alpha = f$  and  $\phi = f_z/(a(f) + 1)dz$ . Then, for any  $c_1$  satisfying  $c_1 < 0$  or  $c_1 > 9/8$ , and any  $c_2 \in R$ , these  $\alpha$ ,  $a(f)$ ,  $c(f)$ , and  $\phi$  satisfy the structure equations (2.1)–(2.6) with  $b = 1$ , and  $\rho = -3$ . Hence, they define a two-parameter family of parallel mean curvature vector surfaces in  $\overline{M}[-12]$  with  $a \neq \bar{a}$  and  $c \neq 0$  such that the Kaehler angle is  $f$ . In particular, if we change the value of  $c_2$  under a fixed  $c_1$ , then the resulting surfaces are isometric and have the same length for the mean curvature vectors. Hence, these give us the associated family of parallel mean curvature vector surfaces in  $\overline{M}[-12]$ .

We note that the case of  $c_1 = 0$  in the example above is already found in Hirakawa [4].

## 5 Appendix

$$\begin{aligned}
t_1(\alpha, a, \bar{a}) &= (-4b + 12b \sin^2 \alpha + 4a + 3a \sin^2 \alpha) \frac{\cot \alpha}{-2 + 3 \sin^2 \alpha}, \\
t_2(\alpha, a, \bar{a}) &= 2a(\bar{a} - b) \cot \alpha + \frac{3}{2} \rho \sin \alpha \cos \alpha, \\
t_3(\alpha, a, \bar{a}) &= -t_1 t_2 - t_2(a - b) \cot \alpha + 3a(\bar{a} - b) t_1 \cot \alpha - a(\bar{a} + b) \frac{\partial t_1}{\partial \alpha} - a t_2 \frac{\partial t_1}{\partial a} \\
&\quad + (a + b) \frac{\partial t_2}{\partial \alpha} + \bar{t}_2 \frac{\partial t_2}{\partial \bar{a}}, \\
t_4(\alpha, a, \bar{a}) &= 2 \left( t_2 \cot \alpha - (a - b)(\bar{a} - b) \cot^2 \alpha - \frac{(a - b)(\bar{a} + b)}{\sin^2 \alpha} \right) + \bar{t}_1(a - b) \cot \alpha \\
&\quad - (a + b) \frac{\partial \bar{t}_1}{\partial \alpha} - \bar{t}_2 \frac{\partial \bar{t}_1}{\partial \bar{a}}, \\
t_5(\alpha, a, \bar{a}) &= t_3 \bar{a} - \bar{t}_4 \left( a \bar{a} + \frac{\rho}{2} (-2 + 3 \sin^2 \alpha) \right), \\
t_6(\alpha, a, \bar{a}) &= \frac{-t_5}{\rho/2(-2 + 3 \sin^2 \alpha)} (a \bar{a} + \frac{\rho}{2} (-2 + 3 \sin^2 \alpha)), \\
t_7(\alpha, a, \bar{a}) &= -\bar{t}_3 + \frac{\partial t_6}{\partial a}, \\
t_8(\alpha, a, \bar{a}) &= -3t_6(a - b) \cot \alpha + (a + b) \frac{\partial t_6}{\partial \alpha} + a t_1 \frac{\partial t_6}{\partial a} + \bar{t}_2 \frac{\partial t_6}{\partial \bar{a}}, \\
t_9(\alpha, a, \bar{a}) &= t_6 \left( -(\bar{a} - b) t_7 \cot \alpha + (\bar{a} + b) \frac{\partial t_7}{\partial \alpha} + t_2 \frac{\partial t_7}{\partial a} + \bar{a} t_1 \frac{\partial t_6}{\partial \bar{a}} \right) - t_7 \bar{t}_8, \\
t_{10}(\alpha, a, \bar{a}) &= t_6 \left( t_3 t_7 - t_7 \bar{t}_7 - 4(\bar{a} - b) t_8 \cot \alpha + (\bar{a} + b) \frac{\partial t_8}{\partial \alpha} + t_2 \frac{\partial t_8}{\partial a} + \bar{a} t_1 \frac{\partial t_8}{\partial \bar{a}} \right) - t_8 \bar{t}_8
\end{aligned}$$



$$\begin{aligned}
& -t_6^2 \left( 3\overline{t_2} \cot \alpha - 3 \frac{(\bar{a} - b)(a + b)}{\sin^2 \alpha} - 3(\bar{a} - b)(a - b) \cot^2 \alpha + \frac{\partial t_3}{\partial \bar{a}} - \frac{\partial t_7}{\partial \bar{a}} \right), \\
t_{11}(\alpha, a, \bar{a}) &= \frac{1}{2t_9} \left( -t_{10} \pm (t_{10}^2 - 4|t_9|^2 t_6)^{1/2} \right), \\
t_{12}(\alpha, a, \bar{a}) &= t_7 t_{11} + t_8 - t_6 \frac{\partial t_{11}}{\partial a} - \overline{t_{11}} \left( (a + b) \frac{\partial t_{11}}{\partial \alpha} + at_1 \frac{\partial t_{11}}{\partial a} + \overline{t_2} \frac{\partial t_{11}}{\partial \bar{a}} \right), \\
t_{13}(\alpha, a, \bar{a}) &= t_3 + 3(\bar{a} - b)t_{11} \cot \alpha - \left( (\bar{a} + b) \frac{\partial t_{11}}{\partial \alpha} + t_2 \frac{\partial t_{11}}{\partial a} + \overline{t_{11}} \frac{\partial t_{11}}{\partial \bar{a}} + \overline{at_1} \frac{\partial t_{11}}{\partial \bar{a}} \right), \\
\bar{t}_i(\alpha, a, \bar{a}) &= t_i(\alpha, \bar{a}, a) \quad (i = 1, 2, 3, \dots).
\end{aligned}$$

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